Components of Hilbert schemes of points

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1 Introduction

Everything today will be over a field k, which will usually be algebraically closed, and sometimes of characteristic not 2 or 3.

The main object of study today will be the Hilbert scheme of n points in $\mathbb{A}^d$, $\mathrm{Hilb}^n(\mathbb{A}^d)$, which represents the functor

$$\begin{aligned} \mathrm{Sch}/k &\to \mathbf{Set} \\ T &\mapsto \{Y \overset{\mathrm{cl}}{\subset} \mathbb{A}^d_k \times T : \mathcal{O}_Y \text{ is locally free over } \mathcal{O}_T \text{ with rank } n\}. \end{aligned}$$

In particular, we have identifications

$$\begin{aligned} \{k-\text{points of } \mathrm{Hilb}^n(\mathbb{A}^d)\} &\leftrightarrow \{\text{length-}n \text{ closed subschemes of } \mathbb{A}^d_k\} \\ &\leftrightarrow \{\text{ideals } I \subset S := k[x_1, \ldots, x_d] \text{ with } \dim_k S/I = n\} \end{aligned}$$

Question 1. What can we say about the irreducible components of $\mathrm{Hilb}^n(\mathbb{A}^d)$? In particular, when is it irreducible?

Aside: Suppose $k = \overline{k}$, and X is a smooth connected k-variety with dimension d. I claim that the question of irreducibility of $\mathrm{Hilb}^n(X)$ is equivalent to that of $\mathrm{Hilb}^n(\mathbb{A}^d)$. Reason: as I'll explain in a bit, we can reduce to studying local algebras; i.e. algebras supported at some closed point $x \in X$. But on some open neighborhood U of x, we can choose coordinates defining an étale map $\varphi : U \to \mathbb{A}^d$. This is formally locally an isomorphism near x, so it induces a bijection between the closed subschemes of X supported at x and the closed subschemes of $\mathbb{A}^d$ supported at $\varphi(x)$.

Some simple cases where it is irreducible:

*Notes for a talk in Berkeley's student arithmetic geometry seminar. References: Dustin Cartwright, Daniel Erman, Mauricio Velasco, and Bianca Viray, "Hilbert schemes of 8 points"; Diane Maclagan, "Notes on Hilbert schemes"; Mark Haiman and Bernd Sturmfels, "Multigraded Hilbert schemes"; Franco Rota, "Symmetric products and the Hilbert scheme of points"; Anthony Iarrobino, "Reducibility of the Families of 0-Dimensional Schemes on a Variety" and "Compressed algebras and components of the punctual Hilbert scheme".

Example 2. Take $n = 2$. Geometrically, a length-2 closed subscheme of $\mathbb{A}^d$ can consist either of two distinct (unordered) points or one point equipped with a tangent direction. So we have $\mathrm{Hilb}^2(\mathbb{A}^d) = \mathrm{Bl}_\Delta \mathrm{Sym}^2(\mathbb{A}^d)$. (In fact this is true with $\mathbb{A}^d$ replaced by any smooth variety X/k.) In particular, although there are "more closed subschemes than expected" supported at a point, they are all limits of algebras consisting of two points.

Example 3. Recall from Martin's talk that for any smooth surface[1] X (e.g. $X = \mathbb{A}^2$), $\mathrm{Hilb}^n(X)$ is irreducible, smooth, and birational to $\mathrm{Sym}^n(X)$.

In general, we have a map $\pi : \mathrm{Hilb}^n(\mathbb{A}^d) \to \mathrm{Sym}^n(\mathbb{A}^d)$ sending a length-n subscheme to the list of points (with multiplicity) where it is supported. This map is an isomorphism over the locus $U \subset \mathrm{Sym}^n(\mathbb{A}^d)$ where all the points are distinct. It follows that $\pi^{-1}(U)$ is open in $\mathrm{Hilb}^n(\mathbb{A}^d)$, and in particular is dense in its irreducible component.

Definition 4. We call $\pi^{-1}(U)$ the *smooth locus* (it parametrizes smooth subschemes of dimension 0). We call its closure the *smoothable locus*.

Remarks:

1. The smooth locus is smooth of dimension $n \cdot d$.

2. Since the smoothable locus is an irreducible component of $\mathrm{Hilb}^n(\mathbb{A}^d)$, it follows that $\mathrm{Hilb}^n(\mathbb{A}^d)$ is irreducible if and only if it equals the smoothable locus.

3. Note that $\mathrm{Hilb}^n(\mathbb{A}^d)$ is of finite type over k, since $\mathrm{Hilb}^n(\mathbb{A}^d)$ is open in $\mathrm{Hilb}^n(\mathbb{P}^d) = \mathrm{Quot}^n_{\mathcal{O}_{\mathbb{P}^d}/\mathbb{P}^d/k}$, which is projective. In particular, its closed points ($= k-$points if $k = \overline{k}$) are dense. So for $k = \overline{k}$, $\mathrm{Hilb}^n(\mathbb{A}^d)$ is irreducible if and only if all length-n subschemes of $\mathbb{A}^d$ are smoothable.

4. One can show that a closed subscheme is smoothable if and only if its connected components are. So we can restrict our search for non-smoothable algebras to the local case.

We'll start with two examples of Hilbert schemes of points that fail to be irreducible for two different reasons. Then we'll say something about how to understand the irreducible components of $\mathrm{Hilb}^n(\mathbb{A}^d)$ more generally.

2 Example: 102 points in $\mathbb{A}^3$

In this and the next example, k is an arbitrary field.

Apologies in advance for the numbers. The numbers aren't that important, of course—they are simply chosen to be big enough to work.

Theorem 5. *(Iarrobino) The Hilbert scheme* $\mathrm{Hilb}^{102}(\mathbb{A}^3)$ *is reducible.*

[1]In his talk it was stated only for smooth projective surfaces, but passing to an open subscheme of X doesn't break anything.

Proof. The dimension of the smoothable locus is $102 \cdot 3 = 306$. If we can exhibit a family of distinct length-102 subschemes of $\mathbb{A}^3$ of dimension > 306, this will imply that not all of them are smoothable, so $\mathrm{Hilb}^{102}(\mathbb{A}^3)$ is reducible.

Note that the space S_7 of degree-7 homogeneous polynomials in 3 variables has dimension $\binom{9}{2} = 36$. Our family is parametrized by the Grassmannian of 18-dimensional subspaces of S_7:

$$\mathrm{Gr}(18, S_7) \to \mathrm{Hilb}^{102}(\mathbb{A}^3)$$
$$V \mapsto I = (V, \text{ all octics}).$$

Note that the length of S/I is the number of monomials we haven't modded out:

$$1 + 3 + 6 + 10 + 15 + 21 + 28 + 18 = 102,$$

and the dimension of the Grassmannian is $18 \cdot (36 - 18) = 324 > 306$. □

Why did this work? Of course we can do a similar construction using degree-m polynomials instead of degree-7. The dimension of the space of homogeneous degree-m polynomials is quadratic in m, so the length n of the resulting algebras will be cubic in m. If we choose subspaces $V \subset S_m$ with about half the maximal dimension, then the dimension of the Grassmannian will be quartic in m. So for sufficiently large m, we will have a family of dimension $\gg 3n$.

More generally, if we fix d and let n vary, Iarrobino exhibits a family of local algebras with dimension $\geq cn^{2-2/d}$, which for $3 \leq d \ll n$ exceeds the dimension nd of the smoothable locus.

Exercise 6. Show that we can reduce the number 102 in the example to 96. In fact Iarrobino later showed how to bring it down to 78.

3 Example: 8 points in $\mathbb{A}^4$

We just discussed a Hilbert scheme of points that fails to be irreducible because it has a component that's too big. Now let's see a sort of counterpoint to this: a Hilbert scheme of points that fails to be irreducible because it has a component that's too small.

Theorem 7. *(Iarrobino-Emsalem, Cartwright-Erman-Velasco-Viray) There exists a non-smoothable closed subscheme $Y \hookrightarrow \mathbb{A}^4$ of length 8. In particular, the Hilbert scheme $\mathrm{Hilb}^8(\mathbb{A}^4)$ is reducible.*

Remark 8. In fact the same argument will work for all $n \geq 8$ and $d \geq 4$. Namely, if $d > 4$, just consider $Y \hookrightarrow \mathbb{A}^4 \hookrightarrow \mathbb{A}^d$. If $n > 8$, just add more points, and recall that $Y \hookrightarrow \mathbb{A}^d$ is smoothable if and only if its connected components are.

To prove this, it suffices to exhibit one point on $\mathrm{Hilb}^8(\mathbb{A}^4)$ that does not lie on the smoothable locus. How can one prove that a point isn't in the smoothable locus? Recall two facts:

Fact 1: the dimension of the tangent space is upper semicontinuous.

On the *smooth* locus (i.e. the locus of $\mathrm{Hilb}^8(\mathbb{A}^4)$ parametrizing choices of eight geometrically distinct points of $\mathbb{A}^4$), the tangent space has dimension $8 \cdot 4 = 32$. So if we can find a point whose tangent space has dimension less than 32, it must not be in the smoothable locus.

Fact 2: we can calculate the dimension of the tangent space with the following formula from Martin's 9/17/21 talk:

Proposition 9. *The tangent space to* $\mathrm{Hilb}_{X/k}$ *at the* k*-point specified by* $Y \hookrightarrow X$ *is*

$$\begin{aligned} T_{[Y]} \mathrm{Hilb}_{X/k} &= \mathrm{Hom}_{\mathcal{O}_X}(J_Y, \mathcal{O}_Y \cdot \varepsilon) \\ &= \mathrm{Hom}_{\mathcal{O}_X}(J_Y/J_Y^2, \mathcal{O}_Y) \\ &= H^0(Y, \mathcal{N}_{Y/X}). \end{aligned}$$

So it suffices to write down an ideal $J \subset S = k[x_1, x_2, x_3, x_4]$ such that $\dim_k(S/J) = 8$ and $\dim_k \mathrm{Hom}_S(J, S/J) < 32$.

Here it is: let

$$J = (x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_3x_4, x_1x_4 + x_2x_3) \subset S = k[x_1, x_2, x_3, x_4].$$

Note that S/J has k-dimension 8; a basis is given by $\{1, x_1, x_2, x_3, x_4, x_1x_3, x_1x_4, x_2x_4\}$. So it represents a k-point of $\mathrm{Hilb}^8(\mathbb{A}^4)$. We then calculate that an arbitrary element of $\mathrm{Hom}_S(J, S/J)$ can be written as a matrix of the form

$$\begin{array}{c} \\ 1 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_1x_3 \\ x_1x_4 \\ x_2x_4 \end{array} \begin{array}{c} \begin{array}{ccccccc} x_1^2 & x_1x_2 & x_2^2 & x_3^2 & x_3x_4 & x_4^2 & x_1x_4 + x_2x_3 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2a_2 & a_1 & 0 & 0 & 0 & 0 & a_4 \\ 0 & a_2 & 2a_1 & 0 & 0 & 0 & a_3 \\ 0 & 0 & 0 & 2a_3 & a_4 & 0 & a_1 \\ 0 & 0 & 0 & 0 & a_3 & 2a_4 & a_2 \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{pmatrix} \end{array}$$

where each a_i and each $*$ represents an independent element of k. So we have $\dim_k \mathrm{Hom}_S(J, S/J) = 4 + 21 = 25 < 32$. Thus the point represented by J does not lie on the smoothable component, proving the theorem.

4 How to find the components more generally

Now let k be algebraically closed of characteristic not 2 or 3.

The goal of this section is to shed more light on the previous example by studying the irreducible components of $\mathrm{Hilb}^8(\mathbb{A}^d)$ more carefully. In particular, we will roughly outline the proof of the following theorem:

Theorem 10. *(Cartwright-Erman-Velasco-Viray) If* $d \geq 4$*, then* $\mathrm{Hilb}^8(\mathbb{A}^d)$ *has exactly two irreducible components, namely the smoothable component of dimension* $8d$ *and another component of dimension* $8d-7$*. On the other hand,* $\mathrm{Hilb}^n(\mathbb{A}^d)$ *is irreducible for* $n < 8$ *and all* d*, and for* $n = 8$ *when* $d < 4$*.*

As before, we will focus our attention on local algebras. We will first break the problem into finitely many smaller problems as follows.

Standing notation: $H_n^d := \mathrm{Hilb}^n(\mathbb{A}^d)$.

Definition 11. Given a local algebra S/J, let $\mathfrak{m}$ be its maximal ideal. The *Hilbert function* $\vec{h} = (h_0, h_1, \dots)$ of S/J is given by

$$h_i = \dim_k \mathfrak{m}^i/\mathfrak{m}^{i+1}.$$

Note that we have $\sum_i h_i = n$.

Example 12. The algebra S/J from the example earlier has $\mathfrak{m} = (x_1, x_2, x_3, x_4)$. Its Hilbert function is $(1, 4, 3)$, since it has a k-basis consisting of $1, 4$, and 3 elements that are homogeneous of degrees $0, 1$, and 2 respectively. Note that although this particular ideal is homogeneous, the Hilbert function makes sense even for inhomogeneous ideals.

Definition 13. If $\vec{h}$ is a vector with $\sum_i h_i = n$, let $H_{\vec{h}}^d \subset H_n^d$ be the subset consisting of local algebras supported at the origin with Hilbert function $\vec{h}$. This is a locally closed subset, and we endow it with the reduced subscheme structure.

The key technical tool in the proof is a variant of $H_{\vec{h}}^d$ that only allows homogeneous ideals, introduced by Mark Haiman and Bernd Sturmfels:

Definition 14. The standard graded Hilbert scheme $\mathcal{H}_{\vec{h}}^d$ is the scheme representing the functor:

$$\begin{aligned} &k-\mathbf{alg} \to \mathbf{Set} \\ &\quad T \mapsto \big\{\text{homogeneous ideals } J \subset S \otimes T \text{ such that } (S \otimes T/J)_i \\ &\qquad\quad \text{is a locally free } T\text{-module of rank } h_i \text{ for each } i\big\} \end{aligned}$$

Theorem 15. *(Haiman-Sturmfels) This exists and can be described by explicit equations. (Much more general—they allow "multigradings" by an arbitrary abelian group, arbitrary Hilbert functions, and work over an arbitrary ground ring, which need not even be noetherian!)*

The functorial description gives a natural map $\mathcal{H}_{\vec{h}}^d \to H_n^d$ when $|\vec{h}| = n$; this turns out to be a closed embedding. We also have a map $H_{\vec{h}}^d \to \mathcal{H}_{\vec{h}}^d$ given by sending an algebra to its associated graded algebra. (This is not compatible with the inclusion $H_{\vec{h}}^d \hookrightarrow H_n^d$, but that's fine.)

Then we can understand the standard graded Hilbert schemes $\mathcal{H}_{\vec{h}}^d$ explicitly in terms of Grassmannians. For example, let's consider $\mathcal{H}_{(1,4,3)}^4$, which accounts for[2] the second irreducible component of H_8^4. Recalling that the space S_2 of homogeneous quadratics in $S = k[x_1, x_2, x_3, x_4]$ has dimension $\binom{5}{2} = 10$, we have:

[2] Algebras in $\mathcal{H}_{(1,4,3)}^4$ are by definition supported at the origin, but you can of course translate them elsewhere in $\mathbb{A}^4$.

$$\mathrm{Gr}(10-3, S_2) \cong \mathcal{H}^4_{(1,4,3)}$$
$$V \subset S_2 \leftrightarrow I = (V, \text{ all cubics})$$

Proof outline of Theorem 10:

1. List all irreducible components of standard graded Hilbert schemes $\mathcal{H}^d_{\vec{h}}$ with $|\vec{h}| \leq 8$, by writing the schemes down explicitly in terms of Grassmannians.[3] (If we only care about the irreducible components, then we can even assume $d = h_1$ without loss of generality.)

2. List all irreducible components of $H^d_{\vec{h}}$, by studying the fibers of the maps $\pi_{\vec{h}} : H^d_{\vec{h}} \to \mathcal{H}^d_{\vec{h}}$.

3. For each irreducible component H of each $H^d_{\vec{h}}$, do one of the following:

   (a) Show that a dense subset of the k-algebras in H are isomorphic to each other, and show directly that they are smoothable.

   (b) Find a smooth point of H^d_n contained in H whose corresponding algebra is smoothable. Since it's a smooth point, it must belong to a unique irreducible component of H^d_n, namely the smoothable component, which must therefore contain all of H.

   (c) For $\vec{h} = (1, 4, 3)$, find a point of $H^d_{\vec{h}}$ that is not smoothable. (We did this.)

(In step 1, everything is irreducible except when $\vec{h} = (1, 3, 2, 1)$ or $(1, 4, 2, 1)$, in which case there are two irreducible components. In step 2, we also get two components for $\vec{h} = (1, 3, 2, 1, 1)$.)

This strategy has been pushed further in dimension 3:

Theorem 16. *(Various authors, most recently including Theodosios Douvropoulos, Joachim Jelisiejew, Bernt Ivar Utstøl Nødland, and Zach Teitler) Suppose $k = \overline{k}$ has characteristic 0. For all $n \leq 11$, $\mathrm{Hilb}^n(\mathbb{A}^3)$ is irreducible.*

5 Open questions

1. Let $11 < n < 78$. Is $\mathrm{Hilb}^n(\mathbb{A}^3)$ irreducible?

2. Let $10 \leq n \leq 11$, and let $k = \overline{k}$ of positive characteristic. Is $\mathrm{Hilb}^n(\mathbb{A}^3)$ irreducible?

3. Is Theorem 10 still true in characteristics 2 and 3?

4. Is $\mathrm{Hilb}^8(\mathbb{A}^4)$ reduced?

5. Does any Hilbert scheme of points have a generically non-reduced component?

[3] The reason for the condition $\mathrm{char}\, k \neq 2, 3$ is buried inside this step. Namely, let $S^\vee$ be the graded ring $k[\frac{\partial}{\partial x_i}]$. If $\mathrm{char}\, k \nmid m!$, then $S^\vee_m$ is dual to S_m in the obvious way. We can use this to parametrize the homogeneous ideals in $\mathcal{H}^d_{\vec{h}}$, by saying that certain graded pieces are orthogonal to certain subspaces of $S^\vee_m$.